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On the Markov Inequality in L^P-Spaces

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This paper gives new admissible values for the constant in Markov inequality in the p-metric. We improve a classical theorem of Hille, Szegö, and Tamarkin. For $p=2$, sharp numerical values are obtained. © 1990 Academic Press, Inc.

1. INTRODUCTION

Let H_n be the space of algebraic polynomials of degree at most n, and $\|\cdot\|_p$ the usual p-norm on $[-1, 1]$.

It is well known (see, for example, [4, p. 141]) that, for any $P \in H_n$, the Markov inequality

$$\|P'\|_\infty \leqslant n^2 \|P\|_\infty \tag{1}$$

holds and is optimal since we have equality for the Tchebicheff polynomials.

Inequality (1) has been extended to the p-norm ($p \geqslant 1$) by Hille, Szegö, and Tamarkin [3]. Their result reads

$$\|P'\|_p \leqslant C(n, p) n^2 \|P\|_p, \tag{2}$$

where $C(n, 1) = 2(1+(1/n))^{n+1}$ and $C(n, p) = 2(p-1)^{(1/p)-1}(p+(1/n))[1+p/(np-p+1)]^{n-1+1/p}$ $(p>1)$. Let us note that $C(n, p)$ is a bounded coefficient: $C(n, p) \leqslant 6e^{1+(1/e)}$ $(n>0\ \ p \geqslant 1)$. Furthermore exponent 2 is sharp as can be seen by taking $P = P_n^{(2,2)}$ (Jacobi's polynomials in the ultraspherical case).

We observe that

$$C(n, p) \to 2(1+1/(n-1))^{n-1} < 2e \qquad (n \text{ fixed}, p \to \infty)$$

$$C(n, 1) \to 2e \qquad (n \to \infty)$$

$$C(n, p) \to 2ep(p-1)^{(1/p)-1} \qquad (p>1 \text{ fixed}, n \to \infty).$$

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The purpose of this paper is to give new admissible values for $C(n, p)$ and some computational results for the special case $p=2$.

2. The Case $p=1$

Theorem 1. *For any $P \in H_n$ $(n \geqslant 0)$ we have*

$$\|P'\|_1 \leqslant (8/\pi)^{1/2} (n+3/4)^2 \|P\|_1.$$

This result is quite an improvement of (2) since $(8/\pi)^{1/2} = 1.5957691...$ while in (2) $C(n, 1)$ lies between $2e$ $(=5.436...)$ and 8.

Before proving Theorem 1 we need some lemmas.

Lemma 1. *For any $\theta \in \left]0, \frac{1}{3}\right]$ we have* $1/\sin\theta \leqslant 1/\theta + \frac{9}{53}\theta$.

Proof. For any $\left]0, \frac{1}{3}\right]$ we have $1/\sin\theta \leqslant 1/(\theta - \theta^3/6)$ and for $\theta > 0$, $1/(\theta - \theta^3/6) \leqslant 1/\theta + \frac{9}{53}\theta$ is equivalent to $\theta^2 \in \left]0, \frac{1}{9}\right]$.

Lemma 2. *For any $x \in \left]0, \frac{1}{3}\right]$ we have*

$$\int_0^x (\theta/\sin\theta)\, d\theta \leqslant x + \tfrac{3}{53}x^3.$$

Proof. From Lemma 1 $\theta/\sin\theta \leqslant 1 + \frac{9}{53}\theta^2$. Integrating both sides of this inequality from 0 to x gives the result.

We denote by S_n the set of trigonometric polynomials of order at most n and by $\|\cdot\|_p^*$ the usual norm in $L^p(0, 2\pi)$.

Lemma 3. *For any $T \in S_n$ we have*

$$\|T'\|_\infty^* \leqslant 1/(2\pi) n(n+1) \|T\|_1^*.$$

Proof. Let us denote by D_n the nth Dirichlet kernel. $D_n(x) = \sum_{-n}^{n} e^{ikx}$. For any x, $|D_n'(x)| \leqslant n(n+1)$ and

$$T'(x) = (1/(2\pi)) \int_0^{2\pi} T(t) D_n'(x-t)\, dt.$$

Then

$$|T'(x)| \leqslant (1/(2\pi)) n(n+1) \int_0^{2\pi} |T(t)|\, dt.$$

We recall the Bernstein inequality [6]: for any $p \geqslant 1$ and any $T \in S_n$, $\|T'\|_p^* \leqslant n \|T\|_p^*$.

PROPOSITION 1. *For any $T \in S_n$ $(n \geqslant 3)$ we have*

$$\|T\|_1^* \leqslant (8/\pi)^{1/2}[((n+1)(n+2)+\tfrac{1}{3})^{1/2} + 1/(20n)]\ \|T(\theta)\sin\theta\|_1^*.$$

Proof. Let $d \in]0, \frac{1}{3}]$, $A = [-d, d] \cup [\pi - d, \pi + d]$ and $B = [-\pi/2, 3\pi/2] \backslash A$. Taylor's formula gives $T(\theta) = (1/\sin\theta)\, T(\theta)\sin\theta = (\theta/\sin\theta)\,(T\cdot\sin)'\,(c(\theta))$ for some $c(\theta)$ between 0 and θ (if $\theta = 0$ replace $\theta/\sin\theta$ by 1). Then

$$\begin{aligned}\int_{-d}^{d} |T(\theta)|\, d\theta &\leqslant \|(T\cdot\sin)'\|_\infty^* \int_{-d}^{d} |\theta/\sin\theta|\, d\theta \\ &\leqslant (1/\pi)(n+1)(n+2)\ \|T(\theta)\sin\theta\|_1^* \int_0^d |\theta/\sin\theta|\, d\theta \\ &\qquad \text{(Lemma 3)} \\ &\leqslant (1/\pi)(n+1)(n+2)\ \|T(\theta)\sin\theta\|_1^*\ (d + \tfrac{3}{53}d^3) \\ &\qquad \text{(Lemma 2)}.\end{aligned}$$

A similar calculation gives

$$\int_{\pi-d}^{\pi+d} |T(\theta)|\, d\theta \leqslant (1/\pi)(n+1)(n+2)(d + \tfrac{3}{53}d^3)\ \|T(\theta)\sin\theta\|_1^*$$

and

$$\int_A |T(\theta)|\, d\theta \leqslant (2/\pi)(n+1)(n+2)(d + \tfrac{3}{53}d^3)\ \|T(\theta)\sin\theta\|_1^*. \tag{3}$$

For $x \in B$, $|\sin x| > \sin d$. A use of Lemma 1 yields

$$\int_B |T(\theta)|\, d\theta \leqslant (1/d + \tfrac{9}{53}d)\ \|T(\theta)\sin\theta\|_1^*. \tag{4}$$

Estimates (3) and (4) together give

$$\begin{aligned}\|T\|_1^* &\leqslant [(2/\pi)(n+1)(n+2)(d + \tfrac{3}{53}d^3) \\ &\qquad + (1/d) + \tfrac{9}{53}d]\ \|T(\theta)\sin\theta\|_1^* \\ &\leqslant [((2/\pi)(n+1)(n+2) + \tfrac{9}{53})d + (1/d) \\ &\qquad + (2/\pi)(n+1)(n+2)\tfrac{3}{53}d^3]\ \|T(\theta)\sin\theta\|_1^* \qquad \text{(Lemma 2)}.\end{aligned}$$

We now choose $d = [(2/\pi)(n+1)(n+2) + \frac{9}{53}]^{-1/2}$. Clearly for $n \geqslant 3$ we have $d < \frac{1}{3}$, and the expression between the square brackets becomes

$$\begin{aligned}&[2[(2/\pi)(n+1)(n+2) + \tfrac{9}{53}]^{1/2} + (2/\pi)(n+1)(n+2)\tfrac{3}{53}d^3] \\ &\quad = (8/\pi)^{1/2}\ [[(n+1)(n+2) + (9\pi/106)]^{1/2} \\ &\qquad + (2\pi)^{-1/2}\ (n+1)(n+2)\tfrac{3}{53}d^3] \\ &\quad \leqslant (8/\pi)^{1/2}\ [[(n+1)(n+2) + \tfrac{1}{3}]^{1/2} + 1/(20n)]\end{aligned}$$

due to the facts that $9\pi/106 < \frac{1}{3}$ and

$$(2\pi)^{-1/2}(n+1)(n+2)\tfrac{3}{53}d^3 < (\pi/4)\tfrac{3}{53}(1/n) < 1/(20n).$$

Proof of Theorem 1. A straightforward calculation shows that if $P \in H_1$ then $\|P'\|_1 \leqslant 2\,\|P\|_1$ (with equality for $P(x) = x$), and if $P \in H_2$ then $\|P'\|_1 \leqslant 4\,\|P\|_1$ (with equality for $P(x) = 4x^2 - 1$) so, in the following, we assume $n \geqslant 3$ and we can apply Proposition 1.

Let $P \in H_n$ $(n \geqslant 3)$ and $T(\theta) = P(\cos\theta)$. We have

$$\begin{aligned}\|P'\|_1 &\leqslant \tfrac{1}{2}\,\|T'\|_1^* \leqslant (n/2)\,\|T\|_1^* \qquad \text{(Bernstein inequality)}\\ &\leqslant (n/2)(8/\pi)^{1/2}\,[((n+1)(n+2)+\tfrac{1}{3})^{1/2} + 1/(20n)]\\ &\quad \times \|T(\theta)\sin\theta\|_1^* \qquad \text{(by Proposition 1)}\\ &= (8/\pi)^{1/2}\,[n((n+1)(n+2)+\tfrac{1}{3})^{1/2} + \tfrac{1}{20}]\,\|P\|_1\end{aligned}$$

and it is easy to check that the coefficient between the square brackets is less than $(n+\frac{3}{4})^2$.

3. The Case $p > 1$

Theorem 2. *For any $P \in H_n$ and $p > 1$ we have*

$$\|P'\|_p \leqslant Cn^2\,\|P\|_p,$$

where

$$C = \left[\frac{(2p+1)^{2+1/p}}{p(p+1)}\right]^{(p-1)/(p+1)} \left[2p\,\frac{p+1}{p-1}\right]^{1/p} \left[\frac{p-1}{2}\right]^{2/p(p+1)} \times \left[1 - \frac{3}{5n}\right]^{1-1/p} \left[1 + \frac{1}{np}\right]^{1-1/p}.$$

Despite its complicated expression, coefficient C has some interesting properties:

$C \to 4(1 - 3/5n)$ (instead of $2e$ in (2)) as $p \to \infty$, n being fixed,

C is less than the constant $C(n, p)$ in (2). Examples are shown in Table I.

Lemma 4. *For $x \in [0, 1]$, $\sin x \geqslant x/(1 + x/5)$.*

Proof. For $x \in\,]0, 1]$, $\sin x > x - x^3/6 \geqslant 1/(1/x + \frac{1}{5})$ since the last inequality is equivalent to $x^2 + 5x - 6 \leqslant 0$.

TABLE I

	$P=2$	$P=3$	$P=4$	$P=5$	$P=10$	$P=1000$
$N=2$	$H=10.76$ $C=8.46$	$H=9.30$ $C=7.23$	$H=8.23$ $C=6.39$	$H=7.51$ $C=5.82$	$H=5.92$ $C=4.54$	$H=4.03$ $C=2.84$
$N=5$	$H=10.85$ $C=9.17$	$H=9.91$ $C=8.29$	$H=9.05$ $C=7.52$	$H=8.41$ $C=6.95$	$H=6.91$ $C=5.57$	$H=4.92$ $C=3.57$
$N=20$	$H=10.87$ $C=9.45$	$H=10.19$ $C=8.77$	$H=9.42$ $C=8.05$	$H=8.83$ $C=7.49$	$H=7.37$ $C=6.08$	$H=5.34$ $C=3.93$
$N=50$	$H=10.87$ $C=9.50$	$H=10.24$ $C=8.87$	$H=9.49$ $C=8.15$	$H=8.91$ $C=7.60$	$H=7.46$ $C=6.18$	$H=5.42$ $C=4.01$
$N=100$	$H=10.87$ $C=9.52$	$H=10.26$ $C=8.90$	$H=9.52$ $C=8.19$	$H=8.94$ $C=7.63$	$H=7.49$ $C=6.21$	$H=5.45$ $C=4.03$
$N=200$	$H=10.87$ $C=9.53$	$H=10.27$ $C=8.91$	$H=9.53$ $C=8.21$	$H=8.95$ $C=7.65$	$H=7.51$ $C=6.23$	$H=5.47$ $C=4.04$

Note. $H=C(n,p)$ in (2) by Hille, Szegö, and Tamarkin; $C=$ improved constant in Theorem 2.

LEMMA 5. *For $a\in\,]0,1]$ and $n\in\mathbb{N}^*$ we have*

$$\int_0^{a/n}(\sin\theta)^p\,d\theta>\frac{a^{p+1}}{(p+1)(n+\frac{1}{5})^{p+1}}.$$

Proof. For $x\in[0,1]$, $\sin x\geqslant x/(1+x/5)$ (Lemma 5). Thus $\int_0^x(\sin t)^p\,dt\geqslant\int_0^x[t/(1+t/5)]^p\,dt\geqslant\int_0^x t^p/(1+t/5)^{p+2}\,dt=x^{p+1}/(p+1)(1+x/5)^{p+1}$. Taking $x=a/n$ we get

$$\int_0^{a/n}(\sin\theta)^p\,d\theta\geqslant\frac{a^{p+1}}{(p+1)(n+a/5)^{p+1}}\geqslant\frac{a^{p+1}}{(p+1)(n+\frac{1}{5})^{p+1}}.$$

PROPOSITION 2 [3, p. 733, Lemma 3.1]. *For any $T\in S_n$ we have*

$$\int_0^{2\pi}|T(\theta)|^p\,d\theta\leqslant C_1(n,p)\,n\int_0^{2\pi}|T(\theta)|^p\,|\sin\theta|\,d\theta$$

with $C_1(n,p)=2p(1+1/(np))^{np+1}$.

PROPOSITION 3. *For any $T\in S_n$ we have*

$$\int_0^{2\pi}|T(\theta)|^p\,|\sin\theta|\,d\theta\leqslant C_2(p)(n+\tfrac{2}{5})^{p-1}\,\|T(\theta)\sin\theta\|_p^{*p},$$

where

$$C_2(p) = \left[\frac{(2p+1)^{2p+1}}{p^p(p+1)^p}\right]^{(p-1)/(p+1)} \left[\frac{p+1}{p-1}\right]\left[\frac{p-1}{2}\right]^{2/(p+1)}.$$

Proof. Let $T \in S_n$ and θ_0 be such that $|T(\theta_0)| = \|T\|_\infty^*$. Let $a \in]0, 1]$, $J = [\theta_0 - a/n, \theta_0 + a/n]$. By the Bernstein inequality, for $\theta \in J$, $|T(\theta) - T(\theta_0)| \leqslant |\theta - \theta_0| \; \|T'\|_\infty^* \leqslant a \, \|T\|_\infty^* = a \, |T(\theta_0)|$. Thus $|T(\theta)| \geqslant (1-a) \, \|T\|_\infty^*$.

Furthermore

$$\int_J |T(\theta)|^p \, |\sin\theta|^p \, d\theta \geqslant (1-a)^p \, \|T\|_\infty^{*p} \int_J |\sin\theta|^p \, d\theta$$

$$\geqslant (1-a)^p \, \|T\|_\infty^{*p} \int_{-a/n}^{a/n} |\sin\theta|^p \, d\theta$$

and using Lemma 5

$$\int_0^{2\pi} |T(\theta)|^p \, |\sin\theta|^p \, d\theta \geqslant 2(1-a)^p \, \|T\|_\infty^{*p} \frac{a^{p+1}}{(p+1)(n+\frac{1}{5})^{p+1}}.$$

For $b \in]0, 1]$, let $L = [-b/(n+\frac{1}{5}), b/(n+\frac{1}{5})] \cup [\pi - b/(n+\frac{1}{5}), \pi + b/(n+\frac{1}{5})]$. We have $\int_L |T(\theta)|^p \, |\sin\theta| \, d\theta \leqslant 4 \, \|T\|_\infty^{*p} \int_0^{b/(n+1/5)} \sin\theta \, d\theta \leqslant [2b^2/(n+\frac{1}{5})^2] \, \|T\|_\infty^{*p}$ and if $\theta \in [-\pi/2, 3\pi/2] \setminus L$ then $|\sin\theta| > \sin(b/(n+\frac{1}{5})) \geqslant b/(n+\frac{2}{5})$ (Lemma 4). Then

$$\int_0^{2\pi} |T(\theta)|^p \, |\sin\theta| \, d\theta$$

$$\leqslant \left[\frac{b^2(p+1)}{a^{p+1}(1-a)^p} + \frac{1}{b^{p-1}}\right] (n+\tfrac{2}{5})^{p-1} \, \|T(\theta)\sin\theta\|_p^{*p}.$$

In order to minimize the coefficient between the square brackets we choose $a = (p+1)/(2p+1)$,

$$b = 2^{-1/(p+1)} \left[\frac{p-1}{p+1}\right]^{1/(p+1)} a(1-a)^{p/(p+1)},$$

and this coefficient becomes

$$\left[\frac{(2p+1)^{2p+1}}{p^p(p+1)^p}\right]^{(p-1)/(p+1)} \left[\frac{p+1}{p-1}\right]\left[\frac{p-1}{2}\right]^{2/(p+1)}.$$

Proof of Theorem 2. Let $P \in H_n$ $(n \geqslant 1)$ and $T(\theta) = P(\cos\theta)$. We have

$$\begin{aligned}
\|P'\|_p^p &= \tfrac{1}{2}\int_0^{2\pi} |P'(\cos\theta)|^p\,|\sin\theta|\,d\theta \\
&\leqslant \tfrac{1}{2}C_2(p)(n-\tfrac{3}{5})^{p-1}\int_0^{2\pi} |T'(\theta)|^p\,d\theta && \text{(by Proposition 3)} \\
&\leqslant \tfrac{1}{2}C_2(p)\,n^p(n-\tfrac{3}{5})^{p-1}\,\|T\|_p^{*p} && \text{(by Bernstein inequality)} \\
&\leqslant C_1(n,p)\,C_2(p)\,n^p(n-\tfrac{3}{5})^{p-1}\,n\,\|P\|_p^p && \text{(by Proposition 2).}
\end{aligned}$$

Then

$$\|P'\|_p^p \leqslant C_1(n,p)\,C_2(p)\left(1-\frac{3}{5n}\right)^{p-1} n^{2p}\,\|P\|_p^p. \tag{5}$$

Theorem 2 follows by taking the pth-root of both sides in (5).

4. The Case $p=2$

This case has been investigated many times:

In 1937 Hille, Szegö, and Tamarkin [3] proved that $C(n,2) \to 1/\pi$ as $n \to \infty$ and in 1943 Schmidt [5] gave the following result:

$$\text{for}\quad n \geqslant 5, \qquad C(n,2) = B(n,R) = \frac{1}{\pi}\frac{1}{n^2}\,\frac{(n+\frac{3}{2})^2}{1-\dfrac{\pi^2-3}{12(n+\frac{3}{2})^2}+\dfrac{R}{(n+\frac{3}{2})^4}}, \tag{6}$$

where $-6 < R < 13$.

In 1944 Bellman [1] using a method based on Legendre polynomials proved that $C(n,2) \leqslant 1/\sqrt{2}$. Actually refining his method we can show that $C(n,2) \leqslant \sqrt{\frac{1}{8}(1+1/n)(1+2/n)(1+3/n)}$.

In 1987 Dörfler [2] proved that the exact value for $C(n,2)n^2$ is the square root of the largest eigenvalue of $A_n^t A_n$ where A_n is the matrix $(\int_{-1}^{1} p_i'(x)\,p_j(x)\,dx)_{0 \leqslant i \leqslant n-1,\, 0 \leqslant j \leqslant n}$ and (p_i) is the orthonormal system of Legendre polynomials.

Computation of $C(n,2)$, $n < 66$. The computation is made using Dörfler's method. We recall that classical Legendre polynomials (P_n) satisfy

$$P_n' = (2n-1)P_{n-1} + (2n-3)P_{n-3} + \cdots \qquad \text{and} \qquad \|P_n\|_2 = (n+\tfrac{1}{2})^{-1/2}.$$

The associated orthonormal polynomials (p_n') thus satisfy

$$p_n' = \sqrt{2n+1}\,[\sqrt{2n-1}\,p_{n-1} + \sqrt{2n-3}\,p_{n-3} + \cdots]$$

whence

$$(A_n)_{i,j} = \begin{cases} (2i-1)(2j-1) & \text{if } \; i+j \text{ is odd and } i<j \\ 0 & \text{otherwise.} \end{cases}$$

The calculation of $C(n, 2)$ can be performed by hand for $n<5$: we obtain

$$C(1, 2) = \sqrt{3}, \qquad C(2, 2) = \sqrt{15}/4,$$
$$C(3, 2) = \tfrac{1}{9}\sqrt{\tfrac{1}{2}(45+\sqrt{1605})}, \qquad C(4, 2) = \tfrac{1}{16}\sqrt{\tfrac{1}{2}(105+\sqrt{7245})}.$$

The calculation has been made with a computer for $n \leqslant 65$. Results are shown in Table II.

PROPOSITION 4. *$C(n, 2)$ is a decreasing function of n.*

To prove this, we use (6) and establish that for $n \geqslant 5$, $B(n, 13) > B(n-1, -6)$. The calculation is tedious but not difficult: if we set $x = n + \frac{3}{2}$ we are led to examine the sign of a polynomial of degree 10 which is easily proved to be positive for $x \geqslant 3$.

TABLE II

Optimal Values of $C(n, 2)$, $n < 66$

n	$C(n, 2)$	n	$C(n, 2)$	n	$C(n, 2)$
1	1.732050808	23	0.3615229545	45	0.3399739271
2	0.9682458357	24	0.3596550368	46	0.3394935513
3	0.7246218726	25	0.3579416963	47	0.3390340081
4	0.6093630858	26	0.3563645313	48	0.3385939744
5	0.5436561184	27	0.3549079455	49	0.3381722299
6	0.5016567975	28	0.3535586348	50	0.3377676604
7	0.4726480478	29	0.3523051803	51	0.3373792379
8	0.4514682813	30	0.3511377267	52	0.3370060134
9	0.4353500607	31	0.3500477249	53	0.3366471147
10	0.4226846279	32	0.3490277199	54	0.3363017295
11	0.4124760735	33	0.3480711862	55	0.3359691140
12	0.4040761623	34	0.3471723808	56	0.3356486575
13	0.3970453687	35	0.3463262349	57	0.3353394481
14	0.3910754153	36	0.3455282550	58	0.3350411548
15	0.3859438307	37	0.3347744451	59	0.3347531293
16	0.3814861142	38	0.3440612383	60	0.3344748503
17	0.3775780537	39	0.3433854430	61	0.3342058307
18	0.3741241300	40	0.3427441899	62	0.3339456179
19	0.3710496983	41	0.3421349011	63	0.3336937833
20	0.3682955944	42	0.3415552445	64	0.3334499324
21	0.3658143221	43	0.3410031092	65	0.3332136888
22	0.3635673197	44	0.3404765844		

COROLLARY. *For* $n > 64$, $1/\pi < C(n, 2) < \frac{1}{3}$.

Open Problem. $C(n, 2)$ is a decreasing function of n. Is it the same for $C(n, p)$ $p \neq 2$? If the answer were affirmative the estimate $\|P'\|_p \leq (p+1)^{1/p} n^2 \|P\|_p$ should be true since for any p, $C(1, p) = (p+1)^{1/p}$.

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